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LETTER TO THE EDITOR

On the class of possible non-local anyon-like operators and quantum groups

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Abstract. We find a class of non-local operators constructed by attaching a disorder operator to fermionic degrees of freedom, which can be used to generate q -deformed algebras following the Schwinger approach. This class includes the recently proposed anyonic operators defined on a lattice.

Quantum groups and quantum algebras [1–3] have attracted a great deal of attention in recent years [4, 5]. In particular, q -oscillators have been formulated [6, 7] and the Jordan–Schwinger approach [8] has been extensively used for the constructions of quantum algebras [6, 7, 9] and quantum superalgebras [10]. On the other hand, anyons [11] (particles with fractional statistics) are of considerable interest since they appear in physical systems. In recent papers [12–15], the construction of anyonic operators on a two-dimensional lattice has been studied. These anyonic oscillators are substantially different from q -oscillators, since the latter are local operators and can be defined in any space dimension, while anyons are intrinsically two-dimensional non-local objects. (For a survey and discussion, see [16].)

The anyonic operators on a lattice have been used to realize explicitly the quantum algebra $SU_q(2)$ [15], as well as other q -deformed Lie algebras [17], by means of a generalized Schwinger construction. The latter construction differs from the one that involves q -oscillators in the sense that non-local operators are used instead of local ones. Therefore, it is of interest to study whether the anyons are the only non-local operators from which q -deformed algebras can be realized and whether there exists a more general class of such operators.

In this letter we find a class of non-local operators, constructed by attaching a disorder operator [12] to each fermionic degree of freedom, from which the q -deformed algebras can be realized following the Schwinger approach. Since this general class includes, as a particular case, the anyonic operators defined on a lattice in [15], we shall call the operators in this class ‘anyon-like’.

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It is well known that one of the simplest forms of the explicit realization of $SU(2)$ algebra is the Schwinger construction [8]. This can be done by introducing a pair of bosonic or fermionic oscillators. However, in the latter case, unlike the bosonic one, only the 0 and $\frac{1}{2}$ representations of $SU(2)$ are recovered. To obtain the full set of representations, we must introduce several copies of fermionic oscillator pairs.

Recently, the Schwinger construction has also been used to realize the so-called q -deformed (quantum) algebras [6, 7, 9] and quantum superalgebras [10]. If we introduce a pair of bosonic q -oscillators a_i , $i=1, 2$, which satisfy the relations,

$$\begin{aligned} a_i a_i^+ - q a_i^+ a_i &= q^{-N_i} & i=1, 2 \\ [a_i, a_j^+] &= 0 & i \neq j \end{aligned} \quad (1)$$

then one can show that the generators defined as

$$J^+ = a_1^+ a_2 \quad J^- = a_2^+ a_1 \quad J^0 = \frac{1}{2}(N_1 - N_2) \quad (2)$$

satisfy the so-called $SU_q(2)$ algebra

$$\begin{aligned} [J^0, J^\pm] &= \pm J^\pm \\ [J^+, J^-] &= [2J^0] \end{aligned} \quad (3)$$

where

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$$

We can use instead an ordered set of fermionic oscillators $c_i(x)$, satisfying the anticommutation relations

$$\{c_i(x), c_j^+(y)\} = \delta_{ij} \delta(x, y) \quad \{c_i(x), c_j(y)\} = 0 \quad (4)$$

$i, j=1, 2$; where x and y belong to some countable, discrete set Ω , with an ordering relation defined on it, and $\delta(x, y)$ is a delta function in Ω^\dagger . Then with the use of the non-cocommutative comultiplication [18], we define the local (as functions of x) generators

$$\begin{aligned} J^\pm(x) &= \prod_{y < x} q^{-j^0(y)} j^\pm(x) \prod_{z > x} q^{j^0(z)} \\ J^0(x) &= j^0(x) \end{aligned} \quad (5)$$

where

$$\begin{aligned} J^+(x) &= c_1^+(x) c_2(x) & J^-(x) &= c_2^+(x) c_1(x) \\ J^0(x) &= \frac{1}{2} \{c_1^+(x) c_1(x) - c_2^+(x) c_2(x)\} \end{aligned} \quad (6)$$

and

$$\begin{aligned} [j^0(x), j^\pm(y)] &= \pm j^\pm(x) \delta(x, y) \\ [j^+(x), j^-(y)] &= 2j^0(x) \delta(x, y). \end{aligned} \quad (7)$$

[†] This set could be a lattice when x and y are space coordinates.

From (5) and (7) it is straightforward to verify that

$$\begin{aligned}
 [J^0(x), J^\pm(y)] &= \pm J^\pm(x)\delta(x, y) \\
 [J^+(x), J^-(y)] &= 0 \quad \text{if } x \neq y \\
 [J^+(x), J^-(x)] &= \prod_{y < x} q^{-2J^0(y)} 2J^0(x) \prod_{z > x} q^{2J^0(z)}
 \end{aligned} \tag{8}$$

and that the global generators defined as

$$J^\pm = \sum_x J^\pm(x) \quad J^0 = \sum_x J^0(x) \tag{9}$$

satisfy the $SU_q(2)$ algebra (3). Notice also that $(J^+(x))^+ \neq J^-(x)$, but we have instead the relation

$$(J^+(x))^+ = J^-(x)|_{q \rightarrow q^{-1}}. \tag{10}$$

The above analysis, concerning the fermionic realization of $SU_q(2)$ algebra, gives rise to the following question: do there exist operators, which can be used to perform the Schwinger construction for the $SU_q(2)$ algebra (3)? In [15] it was shown that anyonic operators on a two-dimensional lattice can be used to build this algebra explicitly. We propose a more general class of non-local operators, which we call ‘anyon-like’ and which can be used in the Schwinger construction. The use of the terminology ‘anyon-like’ will be more clear when we define operators and obtain their commutation relations which are similar to those of anyons.

Let us define the non-local operators $\alpha_i(x), \alpha_i^+(x)$ as

$$\begin{aligned}
 \alpha_i(x) &= q^{-\Delta_i(x)} c_i(x) \\
 \alpha_i^+(x) &= c_i^+(x) q^{\Delta_i(x)}
 \end{aligned} \tag{11}$$

($i = 1, 2$), where $c_i(x), c_i^+(x)$ are the fermionic operators, which satisfy (4) and

$$\begin{aligned}
 \Delta_i(x) &= \sum_y f(x, y) N_i(y) \\
 N_i(y) &= c_i^+(y) c_i(y).
 \end{aligned} \tag{12}$$

x and y belong to some discrete set Ω and $f(x, y)$ is an arbitrary function to be specified later. The construction of the operators (11) can be understood as a kind of Jordan–Wigner transformation [19] of the fermionic operators, i.e. we attach the disorder operators [12] $q^{-\Delta_i(x)}$ to the fermions $c_i(x)$.

From (12) and (4) it follows that

$$\begin{aligned}
 [\Delta_i(x), c_j(y)] &= -\delta_{ij} f(x, y) c_j(y) \\
 [\Delta_i(x), c_j^+(y)] &= \delta_{ij} f(x, y) c_j^+(y) \\
 [\Delta_i(x), N_j(y)] &= 0.
 \end{aligned} \tag{13}$$

Using (4) and (13) we can calculate the commutation relations of the operators $\alpha_i(x), \alpha_i^+(x)$ defined in (11). We have

$$\alpha_i(x) \alpha_i^+(y) + q^{f(y,x) - f(x,y)} \alpha_i^+(y) \alpha_i(x) = \delta(x, y) \tag{14a}$$

$$\alpha_i(x) \alpha_i(y) + q^{f(x,y) - f(y,x)} \alpha_i(y) \alpha_i(x) = 0. \tag{14b}$$

Notice that when $x=y$, from (14) it follows that

$$\begin{aligned} \{\alpha_i(x), \alpha_i^+(x)\} &= 1 \\ \alpha_i^2(x) &= 0 \end{aligned} \tag{15}$$

i.e. the α_i, α_i^+ operators describe hard core objects which obey standard (fermionic) anticommutation relations at the same point. As a special case, the operators α_i, α_i^+ can be identified with the anyonic operators on a lattice recently introduced in the literature [12-15]. Indeed, the latter can be obtained from (11) by taking $q=e^{i\nu\pi}$ and $f(x, y) = -(1/\pi)\theta(x, y)$, where ν is the statistics determining parameter and $\theta(x, y)$ is the lattice angle function [13, 20].

In what follows we shall refer to the operators $\alpha_i(x), \alpha_i^+(x)$ as anyon-like operators, although it is clear that they are actually anyon operators only for the particular choice of q and $f(x, y)$ given above.

Our aim is to find the general class of functions $f(x, y)$ so that the Schwinger construction can be performed using the operators (11). From the analysis of the fermionic realization of the $SU_q(2)$ algebra given earlier and, in particular, from relation (10) it follows that the operators $\alpha_i(x), \alpha_i^+(x)$ are not sufficient to construct the local generators $J^\pm(x)$. In order to construct them, one can introduce an extra pair of operators $\tilde{\alpha}_i(x), \tilde{\alpha}_i^+(x), i=1, 2$, defined as

$$\begin{aligned} \tilde{\alpha}_i(x) &= q^{\tilde{\Delta}_i(x)} c_i(x) \\ \tilde{\alpha}_i^+(x) &= c_i^+(x) q^{-\tilde{\Delta}_i(x)} \end{aligned} \tag{16}$$

with

$$\tilde{\Delta}_i(x) = \sum_y g(x, y) N_i(y) \tag{17}$$

where $g(x, y)$ is a function, whose properties will be determined later. The commutation relations among the new operators (16) can be obtained from (14) and (15), by replacing q by q^{-1} and $f(x, y)$ by $g(x, y)$.

It is worthwhile to give also the commutation relations between the operators (11) and (16). We have

$$\alpha_i(x) \tilde{\alpha}_i^+(y) + q^{-(f(x,y)+g(y,x))} \tilde{\alpha}_i^+(y) \alpha_i(x) = q^{-(\Delta_i(x)+\tilde{\Delta}_i(x))} \delta(x, y) \tag{18a}$$

$$\alpha_i(x) \tilde{\alpha}_i(y) + q^{f(x,y)+g(y,x)} \tilde{\alpha}_i(y) \alpha_i(x) = 0. \tag{18b}$$

Let us now define the operators

$$\begin{aligned} J_a^+(x) &= \alpha_1^+(x) \alpha_2(x) \\ J_a^-(x) &= \tilde{\alpha}_2^+(x) \tilde{\alpha}_1(x) \\ J_a^0(x) &= \frac{1}{2} \{ \alpha_1^+(x) \alpha_1(x) - \alpha_2^+(x) \alpha_2(x) \} \\ &= \frac{1}{2} \{ \tilde{\alpha}_1^+(x) \tilde{\alpha}_1(x) - \tilde{\alpha}_2^+(x) \tilde{\alpha}_2(x) \}. \end{aligned} \tag{19}$$

First notice that from the definitions (11) and (16) it follows that

$$J_a^0(x) = j^0(x) = \frac{1}{2} \{ N_1(x) - N_2(x) \} \tag{20}$$

where $j^0(x)$ is given in (6). Moreover, we can write

$$J_a^+(x) = q^{-f(x,x) + 2\Sigma_y f(x,y)j^0(y)} j^+(x) \tag{21a}$$

$$J_a^-(x) = q^{g(x,x) + 2\Sigma_y g(x,y)j^0(y)} j^-(x) \tag{21b}$$

with $j^\pm(x)$ also defined in (6).

Using relations (7), equations (21) can be rewritten in a more convenient form, namely

$$J_a^+(x) = q^{(2j^0(x)+1)f(x,x)} \prod_{y < x} q^{2f(x,y)j^0(y)} j^+(x) \prod_{z > x} q^{2f(x,z)j^0(z)} \tag{22a}$$

$$J_a^-(x) = q^{(2j^0(x)+1)g(x,x)} \prod_{y < x} q^{2g(x,y)j^0(y)} j^-(x) \prod_{z > x} q^{2g(x,z)j^0(z)}. \tag{22b}$$

Comparing (20) and (22) with the local generators (5), we see that if we define

$$f(x, y) = g(x, y) = \begin{cases} -\frac{1}{2} & y < x \\ 0 & y = x \\ \frac{1}{2} & y > x \end{cases} \tag{23}$$

then $J_a^\pm(x) = J^\pm(x)$, $J_a^0(x) = J^0(x)$, and consequently, the global generators defined as

$$J_a^\pm = \sum_x J_a^\pm(x) \quad J_a^0 = \sum_x J_a^0(x) \tag{24}$$

will satisfy the $SU_q(2)$ algebra (3). It is clear that for any function $f'(x, y) = af(x, y) = ag(x, y)$, where a is a real constant, the operators (24) will satisfy the algebra $SU_{q'}(2)$ with $q' = q^a$.

Let us remark that the choice of the functions $f(x, y)$ and $g(x, y)$ in (23) in order to get the local generators (5) of the $SU_q(2)$ algebra, is unique. In other words, there exists only one pair of operators $\alpha_i(x)$, $\alpha_i^+(x)$ defined by (11) and one pair of operators $\tilde{\alpha}_i(x)$, $\tilde{\alpha}_i^+(x)$ defined by (16), with $f(x, y)$ and $g(x, y)$ given in (23), so that the Schwinger construction can be performed to build the local generators (5) of the $SU_q(2)$ algebra (3). On the other hand, as we shall see below, there exists a more general class of functions $f(x, y)$ and $g(x, y)$, such that the local operators (20) and (21) satisfy the commutation relations (8) and in consequence, the global generators (24) satisfy the $SU_q(2)$ algebra.

From (20) and (21) and using (7) it is straightforward to verify that

$$[J_a^0(x), J_a^\pm(y)] = \pm J_a^\pm(x) \delta(x, y) \tag{25}$$

and

$$\begin{aligned} J_a^+(x)J_a^-(y) - q^{-2(f(x,y)+g(y,x))} J_a^-(y)J_a^+(x) \\ = q^{-(f(x,x)+g(x,x)) + 2\Sigma_z (f(x,z)+g(x,z))J_z^0(z)} 2J_a^0(x)\delta(x, y). \end{aligned} \tag{26}$$

Assume that for $\forall x, y \in \Omega$

$$f(x, y) = -g(y, x). \tag{27}$$

Then from (26) we conclude that

$$[J_a^+(x), J_a^-(y)] = 0 \quad \forall x \neq y \tag{28}$$

and

$$\begin{aligned}
 [J_a^+(x), J_a^-(x)] &= q^{2\Sigma_x(f(x,z)-f(z,x))J_a^0(z)} 2J_a^0(x) \\
 &= \prod_{y < x} q^{2(f(x,y)-f(y,x))J_a^0(y)} 2J_a^0(x) \prod_{z > x} q^{2(f(x,z)-f(z,x))J_a^0(z)}.
 \end{aligned}
 \tag{29}$$

From (29) one sees that if the function $f(x, y)$ satisfies the relation

$$f(x, y) - f(y, x) = \begin{cases} -\beta & \text{for } y < x \\ 0 & \text{for } y = x \\ \beta & \text{for } y > x \end{cases}
 \tag{30}$$

where β is a real constant, and we define $q' = q^\beta$, then (29) has the same form as one of the local generators of the $SU_{q'}(2)$ algebra (compare with the last equation in (8)). Consequently, relations (25), (28) and (29) imply that the global generators (24) satisfy the $SU_{q'}(2)$ algebra (3).

Thus we have found a wide class of non-local operators, namely, the anyon-like operators (11) and (16), which generate the $SU_q(2)$ algebra through the Schwinger construction. Any function $f(x, y)$ satisfying the relation (30), which only restricts the antisymmetric part of f , yields acceptable anyon-like operators. Our construction generalizes the anyon operators of [15], and it would be interesting to consider generalizing the construction even further.

As mentioned above (see our remarks after (15)), a particular choice of q and $f(x, y)$ in (11) gives rise to the anyonic operators defined on a two dimensional lattice [12-15]. For definiteness, we shall assume that the lattice Ω has spacing one. In constructing the anyonic operators, the basic element is the lattice angle function $\theta(x, y)$ [13, 19]. Here we shall use the specific description of $\theta(x, y)$ given in [15]: The lattice Ω is embedded into a lattice Λ with spacing ε . Then to each point $x \in \Omega$ one associates a cut γ_x , made with bonds of the dual lattice Λ' from minus infinity to $x' = x + 0$ along x -axis, with $0 = (\varepsilon/2, \varepsilon/2)$ the origin of Λ' . In the limit $\varepsilon \rightarrow 0$ we can endow the lattice with an ordering and define [15]

$$\theta_{\gamma_x}(x, y) - \theta_{\gamma_y}(y, x) = \pi \quad \text{for } x > y
 \tag{31}$$

where $\theta_{\gamma_x}(x, y)$ is the angle of the point x measured from the point $y' \in \Lambda'$ with respect to a line parallel to the positive x -axis. This definition of $\theta(x, y)$ is not unique since it depends on the choice of the cuts. If we choose now for each point of the lattice a cut δ_x made with bonds of the dual lattice Λ' from plus infinity to $x' = x - 0$ along x -axis, we will have

$$\tilde{\theta}_{\delta_x}(x, y) - \tilde{\theta}_{\delta_y}(y, x) = -\pi \quad \text{for } x > y
 \tag{32}$$

where $\tilde{\theta}_{\delta_x}(x, y)$ is now the angle of x seen from $y' \in \Lambda'$ with respect to a line parallel to the negative x -axis. Also, we have the relation

$$\tilde{\theta}_{\delta_x}(x, y) - \theta_{\gamma_y}(y, x) = 0 \quad \forall x, y \in \Lambda.
 \tag{33}$$

The angle functions $\theta(x, y)$ and $\tilde{\theta}(x, y)$ allow one to distinguish between clockwise and counterclockwise braidings and, in consequence, to introduce two types of anyonic operators $a(x_\gamma)$ and $a(x_\delta)$, where x_γ denotes the point $x \in \Omega$ with its associated cut γ_x .

The latter operators are defined as [15]

$$\begin{aligned}
 a_i(x_\rho) &= K_i(x_\rho) c_i(x) \\
 K_i(x_\rho) &= \exp\left(i\nu \sum_y \theta_{\rho_x}(x, y) N_i(y)\right) \\
 N_i(y) &= c_i^\dagger(y) c_i(y)
 \end{aligned} \tag{34}$$

where $\rho = \gamma_x$ or δ_x , $i = 1, 2$, ν is the statistical parameter and $c_i(x)$ are the fermionic operators defined on Ω , which obey the anticommutation relations (4) with $\delta(x, y) = 1$ if $x = y$, $\delta(x, y) = 0$ if $x \neq y$.

From the definitions (11) and (16) for the $\alpha_i(x)$ and $\tilde{\alpha}_i(x)$ operators, and from the definitions (34) for the anyonic operators $a_i(x_\gamma)$ and $a_i(x_\delta)$, we see that the latter can be obtained from (11) and (16) by assuming

$$\begin{aligned}
 q &= e^{i\nu\pi} \\
 f(x, y) &= -\frac{1}{\pi} \theta_{\gamma_x}(x, y) \\
 g(x, y) &= \frac{1}{\pi} \tilde{\theta}_{\delta_x}(x, y).
 \end{aligned} \tag{35}$$

Now, relations (31)–(33) imply that (27) and (30) also hold, with $\beta = 1$ in (30). Thus, the global generators (24) defined through the local generators (19), where now instead of $\alpha_i(x)$, $\tilde{\alpha}_i(x)$ we have $a_i(x_\gamma)$, $a_i(x_\delta)$, respectively, satisfy the $SU_q(2)$ algebra (3) with $q = e^{i\nu\pi}$.

It is interesting to remark that anyons can consistently be defined also on a line (one-dimensional chain). In that case one can define

$$\theta_{\gamma_x}(x, y) = -\tilde{\theta}_{\delta_x}(x, y) = \begin{cases} \frac{\pi}{2} & \text{for } x > y \\ -\frac{\pi}{2} & \text{for } x < y \end{cases}$$

so that equations (31)–(33) are fulfilled. Then, from (35) we obtain that $f(x, y) = g(x, y) = -\frac{1}{2}$ for $x > y$ and $\frac{1}{2}$ for $x < y$. If we assume now that $f(x, x) = g(x, x) = 0$ (as in (23)) or, exclude the point $y = x$ from the definition of the disorder operators (34), then we will have that the local generators (22) defined through the anyonic operators will coincide with the iterated coproduct (5) and in consequence, the global generators (24) give the $SU_q(2)$ algebra with $q = e^{i\nu\pi}$.

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